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Type-D space–times with a Killing tensor

M Demiański† and M Francaviglia
Institute of Mathematical Physics, University of Turin, Turin, Italy

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Abstract. We investigate conditions under which a vacuum type-D space–time admits a regular S_2 separability structure. We show that all vacuum type-D space–times without acceleration admit such a structure. It turns out that the structure is completely determined by the Killing tensor and the metric.

1. Introduction

The problem of separability of the Hamilton–Jacobi equation for geodesics has a long history. In recent years this problem has become important in general relativity because properties of geodesics can give much information about the local and global structures of space–time.

The problem of separability of the Hamilton–Jacobi equation for geodesics has been investigated by many authors‡ and necessary and sufficient conditions for separability have finally been formulated by Benenti (1975–6, 1977). For a detailed discussion of these problems see the recent review by Benenti and Francaviglia (1980).

Here we review conditions under which a vacuum type-D space–time admits a Killing tensor and investigate the conditions for existence of a regular S_2 separability structure. It turns out that all vacuum type-D space–times without acceleration admit such a tensor, and also a regular S_2 separability structure. We also show that in vacuum type-D space–times all the information necessary to construct a regular separability structure is contained in the metric and the Killing tensor.

In § 2 we briefly recall the theory of regular separability structures. In § 3 we review the conditions under which a vacuum type-D space–time admits a Killing tensor. In § 4 we finally prove that all vacuum type-D space–times without acceleration admit a regular S_2 separability structure.

2. Regular separability structures

The standard procedure to search for geodesics in a given space–time (V_4, g) is to search for solutions of the Hamilton–Jacobi equation§

$$\tfrac{1}{2}g^{ij}\partial_i W \partial_j W - h = 0 \tag{1}$$

where $h \in R$ is a constant.

† Permanent address: Institute of Theoretical Physics, University of Warsaw, Warsaw, Poland.
‡ See Carter (1969), Walker and Penrose (1970), Hughston *et al* (1972), Hughston and Penrose (1973a, b), Sommers (1973), Woodhouse (1975) and Havas (1975).
§ Throughout this paper, Latin indices $i, j, k, \ldots$ run from 0 to 3.

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Equation (1) is a non-linear partial differential equation and general methods for integrating this equation are not known, unless it can be solved by complete separation of variables. This amounts to finding, if possible, a complete integral $W(x^i)$ of the form

$$W(x^i)=W_0(x^0)+W_1(x^1)+W_2(x^2)+W_3(x^3). \tag{2}$$

We say that a space–time (V_4, g) admits a separability structure if in V_4 there exists a set of local coordinates x^i such that equation (1) admits a complete integral of the form (2). Separability structures in V_4, which are equivalence classes of such coordinate systems, can be classified according to the maximal number r $(0\leq r\leq 4)$ of admitted ignorable coordinates which do not appear explicitly in the metric tensor. Latin indices $a, b, c, \ldots (r+1\leq a, b, c \ldots \leq n)$ are reserved for the remaining non-ignorable coordinates. We denote a separability structure with exactly r ignorable coordinates by S_r. Such a separability structure S_r is regular if in some separable coordinate system associated with it none of the metric components g^{aa} $(a=r+1, \ldots, n$; no summation) is zero.

In the theory of separability structures an important role is played by Killing tensors of order two. They are defined as the symmetric solution of the equation

$$\nabla_{(l}K_{ij)}=0 \tag{3}$$

where round brackets denote symmetrisation.

A geometric characterisation of the regular separability structure S_r is given by the following theorem.

Theorem 1. (Benenti 1975–6, Benenti and Francaviglia 1979 (§ 3), 1980.) A space–time (V_4, g) admits a regular S_r separability structure if and only if it admits r commuting Killing vectors $\underset{\alpha}{X}$ $(0\leq\alpha\leq r-1)$ and $4-r$ Killing tensors $\underset{a}{K}$ $(r\leq a\leq 4)$ which are independent and satisfy the following conditions.

(1) In the Lie algebra of Killing vectors and tensors with Schouten–Nijenhuis brackets the following commutation relations hold:

$$[\underset{a}{K}, \underset{b}{K}]=0 \tag{4a}$$

$$[\underset{a}{K}, \underset{\alpha}{X}]=0 \qquad \forall a, b, \alpha. \tag{4b}$$

(2) The Killing tensors $\underset{a}{K}$ have $4-r$ common eigenvectors $\underset{a}{X}$ such that

$$[\underset{a}{X}, \underset{\alpha}{X}]=[\underset{a}{X}, \underset{b}{X}]=0 \qquad \forall a, b, \alpha \tag{5}$$

$$g(\underset{a}{X}, \underset{\alpha}{X})=0 \qquad \forall a, \alpha. \tag{6}$$

The brackets appearing in (4) are defined by

$$\tfrac{1}{2}[K, H]^{ijl}=K^{m(i}\nabla_m H^{jl)}-H^{m(i}\nabla_m K^{jl)} \tag{7}$$

where ∇_m denotes covariant differentiation. From theorem 1 it follows that the metric tensor g always appears among the Killing tensors $\underset{a}{K}$.

The notion of separability structure is not only useful in testing whether or not the Hamilton–Jacobi equation can be solved by the method of complete separation of variables, but also provides a constructive way for searching for coordinates in which the Hamilton–Jacobi equation is separable. Benenti (1975–6, 1977) showed that in a coordinate system such that $\underset{a}{X}=\partial/\partial x^a$, $\underset{\alpha}{X}=\partial/\partial x^\alpha$ the Hamilton–Jacobi equation is separable.

We are interested in finding the class of vacuum type-D space–times which admit a separability structure. It was shown by Kinnersley (1969) that all vacuum type-D space–times possess at least two commuting Killing vectors. Therefore we investigate conditions under which such space–times admit regular S_2 separability structures with two Killing vectors and two Killing tensors. The metric tensor is one of the Killing tensors and we have to search for an independent one.

3. Killing tensors in vacuum type-D space–times

Following Walker and Penrose (1970) let us decompose the Killing tensor K_{ij} into the traceless part P_{ij} and the pure trace part Kg_{ij}, so we have

$$K_{ij}=P_{ij}+Kg_{ij} \tag{8}$$

where

$$P_{ij}g^{ij}=0 \qquad \text{and} \qquad K=\tfrac{1}{4}g^{ij}K_{ij}. \tag{9}$$

From Killing's equation (3) it follows that

$$\nabla_{(l}P_{ij)}+g_{(ij}\nabla_{l)}K=0 \tag{10}$$

and

$$\nabla_jP_i^j+3\nabla_iK=0. \tag{11}$$

From (10) it follows that P_{ij} is a conformal Killing tensor.† It was shown by Walker and Penrose (1970) that every vacuum type-D space–time admits a conformal Killing tensor. Obviously the Killing tensor K_{ij} exists if and only if (11) is satisfied, i.e. if and only if the divergence of P_{ij} is the gradient of a scalar K.

These conditions were first derived by Hughston *et al* (1972) and investigated by Hughston and Sommers (1973a,b) and Sommers (1973) in the more general case of electrovac space–times with Maxwell field of type {1, 1} whose eigenvectors coincide with those of the Weyl tensor.

Here we investigate integrability conditions for equation (11) in vacuum type-D space–times using a direct method. For this purpose, let us introduce a null tetrad $(l_i, n_i, m_i, \bar{m}_i)$, where l_i and n_i are tangential to two shear-free geodesic congruences determined by the Weyl tensor. The conformal Killing tensor P_{ij} can be written in the form

$$P_{ij}=(\psi\bar{\psi})^{-1/3}(l_{(i}n_{j)}+m_{(i}\bar{m}_{j)}) \tag{12}$$

where ψ is the only non-vanishing component of the Weyl tensor in the canonical tetrad.

Using the Newman–Penrose (1962) formalism and taking into account Bianchi identities we get

$$\nabla_jP_i^j=\tfrac{3}{2}(\psi\bar{\psi})^{-1/3}[(\mu+\bar{\mu})l_i-(\rho+\bar{\rho})n_i+(\pi-\bar{\tau})m_i+(\bar{\pi}-\tau)\bar{m}_i]. \tag{13}$$

† A symmetric tensor P_{ij} is a conformal Killing tensor if there exists a vector P_i such that $\nabla_{(l}P_{ij)}=g_{(ij}P_{l)}$.

Equation (11) can now be reduced to the set of equations

$$
\begin{aligned}
2\Delta K &= -(\psi\bar{\psi})^{-1/3}(\mu+\bar{\mu}) \\
2DK &= (\psi\bar{\psi})^{-1/3}(\rho+\bar{\rho}) \\
2\delta K &= (\psi\bar{\psi})^{-1/3}(\bar{\pi}-\tau) \\
2\bar{\delta}K &= (\psi\bar{\psi})^{-1/3}(\pi-\bar{\tau}).
\end{aligned} \tag{14}
$$

Because of the Bianchi identities, taking $K=\frac{1}{2}(\psi\bar{\psi})^{-1/3}+\frac{1}{2}F$, equations (14) are transformed into

$$
\begin{aligned}
\Delta F &= -2(\psi\bar{\psi})^{-1/3}(\mu+\bar{\mu}) \\
DF &= 2(\psi\bar{\psi})^{-1/3}(\rho+\bar{\rho}) \\
\delta F &= \bar{\delta}F = 0.
\end{aligned} \tag{15}
$$

The integrability conditions for (15) are easily obtained by using commutation relations satisfied by Δ, D, δ and $\bar{\delta}$. We find

$$
\begin{aligned}
&\mu\bar{\rho}-\bar{\mu}\rho = 0 \\
&\pi\bar{\pi}-\tau\bar{\tau} = 0 \\
&\delta(\mu+\bar{\mu}) = -(\mu+\bar{\mu})(\bar{\pi}-\tau)+(\mu+\bar{\mu})(\tau-\bar{\alpha}-\beta) \\
&\delta(\rho+\bar{\rho}) = (\rho+\bar{\rho})(\bar{\alpha}+\beta-2\bar{\pi}+\tau).
\end{aligned} \tag{16}
$$

It is easy to check that these conditions are satisfied in the Kerr space–time. In Boyer–Lindquist coordinates the Kerr metric assumes the well known form

$$
\begin{aligned}
\mathrm{d}s^2 = &\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}t^2+\frac{4Mar\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi-\frac{\Sigma}{\Delta}\,\mathrm{d}r^2-\Sigma\,\mathrm{d}\theta^2 \\
&-\sin^2\theta\left(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\mathrm{d}\phi^2
\end{aligned} \tag{17}
$$

where $\Sigma=r^2+a^2\cos^2\theta$ and $\Delta=r^2-2Mr+a^2$. The null tetrad adopted for the two geodesic shear-free congruences is

$$
\begin{aligned}
l^i &= ((r^2+a^2)/\Delta,\ 1,\ 0,\ a/\Delta) \\
n^i &= (r^2+a^2,\ -\Delta,\ 0,\ a)/2\Sigma \\
m^i &= \frac{1}{\sqrt{2}(r+\mathrm{i}a\cos\theta)}(\mathrm{i}a\sin\theta,\ 0,\ 1,\ \mathrm{i}/\sin\theta)
\end{aligned} \tag{18}
$$

and the non-zero spin coefficients are

$$
\begin{aligned}
&\rho=\frac{-1}{r-\mathrm{i}a\cos\theta} &&\mu=-\frac{\Delta}{2\Sigma(r-\mathrm{i}a\cos\theta)} \\
&\pi=\frac{\mathrm{i}a\sin\theta}{\sqrt{2}(r-\mathrm{i}a\cos\theta)^2} &&\beta=\frac{\tan\theta}{2\sqrt{2}(r+\mathrm{i}a\cos\theta)} \\
&\tau=\frac{-\mathrm{i}a\sin\theta}{\sqrt{2}\Sigma} &&\gamma=\mu+\frac{r-M}{2\Sigma} \qquad \alpha=\pi-\bar{\beta}
\end{aligned} \tag{19}
$$

and the only non-zero tetrad component of the Weyl tensor is

$$\psi = -\frac{M}{(r - \mathrm{i}a \cos \theta)^3}. \tag{20}$$

Substituting these explicit forms for the spin coefficients into conditions (16) and remembering that $\delta = m^i \partial/\partial x^i$, it is apparent that they are satisfied.

Using the explicit forms for the spin coefficients for type-D space–times given by Kinnersley (1969), we checked that all vacuum type-D space–times without acceleration satisfy conditions (16) and therefore they possess a Killing tensor K_{ij}. These results are in agreement with those of Hughston and Sommers (1973a, b) and Sommers (1973).

4. Separability structure in vacuum type-D space–times

In this section we shall show that every vacuum type-D space–time without acceleration admits a regular S_2 separability structure and that it is completely determined by the metric and the Killing tensor K_{ij}.

Following Walker and Penrose (1970) the spinor equivalent of P_{ij} can be written as†

$$P_{ij} = P_{AB}\bar{P}_{A'B'} \tag{21}$$

where P_{AB} is a symmetric spinor satisfying the 'Killing spinor equation'

$$\nabla_{A'(A}P_{BC)} = 0. \tag{22}$$

Using P_{AB} one can define a non-vanishing Killing vector ξ by

$$\xi_i = \xi_{AA'} = \nabla^B{}_{A'}P_{AB}. \tag{23}$$

Moreover, it was shown by Sommers (1973) that P_{ij} is Lie-dragged along ξ, i.e.

$$[\xi, P]_{ij} = 0. \tag{24}$$

In the null tetrad the Killing vector ξ_i is expressed, up to a constant factor, as

$$\xi_i = \psi^{-1/3}(-\mu l_i - \rho n_i + \pi m_i + \tau \bar{m}_i). \tag{25}$$

Using (25) it is easy to show that whenever equations (14) are satisfied $\xi^i \nabla_i K = 0$. Therefore we conclude that K_{ij} is also Lie-dragged along ξ, i.e.

$$[\xi, K]_{ij} = 0. \tag{26}$$

In general ξ is a complex Killing vector and because of the linearity of the Killing equations it defines two real Killing vectors, possibly linearly dependent over R. However, in a generic situation these two vectors do not commute. Therefore we will use only the real part of ξ, which for simplicity will be denoted by the same letter ξ.

Following Sommers (1973) we now introduce another real Killing vector

$$\eta_i = K_{ij}\xi^j, \tag{27}$$

which, of course, commutes with ξ. Without loss of generality, we shall assume that ξ and η are independent.

In fact, whenever it is not so one can show that ξ and η are also linearly dependent and, because of the results of Sommers (1973), the space–time in this case admits four

† We use the Battelle convention (Penrose 1968).

Killing vectors. Since the space–time is of type D, two of them commute and those space–times admit an S_2 separability structure with a reducible Killing tensor (Walker and Penrose 1970). Now we have to prove that η commutes with K_{ij}, i.e. that K_{ij} is Lie-dragged along η. From (27) it follows that

$$(\underset{\eta}{£} K_{ij})\xi^j = 0; \tag{28}$$

here $\underset{\eta}{£} K_{ij} = [\eta, K]_{ij}$ denotes the Lie derivative. Since η is a Killing vector we have also

$$\nabla_l(\underset{\eta}{£} K_{ij}) - \underset{\eta}{£}(\nabla_l K_{ij}) = 0. \tag{29}$$

Symmetrising this relation with respect to l, i, j and using (3) we see that

$$\nabla_{(l}(\underset{\eta}{£} K_{ij)}) = 0 \tag{30}$$

so $\underset{\eta}{£} K_{ij}$ is a Killing tensor. Therefore $\underset{\eta}{£} K_{ij}$ has to be a linear combination of K_{ij}, g_{ij} and all possible symmetrised products of Killing vectors. From the fact that in vacuum type-D space–times none of the Killing vectors has a constant norm, and from (28) and (27), it follows that

$$[\eta, K]_{ij} = 0. \tag{31}$$

Let us consider the distribution spanned by the Killing vectors ξ and η. Its orthogonal complementary distribution is spanned by the vectors

$$\begin{aligned} \underset{1}{e}_i &= -(\mu P + \bar{\mu}\bar{P})l_i + (\rho P + \bar{\rho}\bar{P})n_i \\ \underset{2}{e}_i &= (\pi P + \bar{\tau}\bar{P})m_i - (\bar{\pi}\bar{P} + \tau P)\bar{m}_i \end{aligned} \tag{32}$$

where $P = \psi^{-1/3}$.

It can be easily checked using (8) and (12) that $\underset{1}{e}$ and $\underset{2}{e}$ determine two eigendirections of the Killing tensor. Using the Newman–Penrose (1962) formalism it is possible to show by a long calculation that in type-D space–times without acceleration it is always possible to scale $\underset{1}{e}$ and $\underset{2}{e}$ so that they commute. Of course, $\underset{1}{e}$ and $\underset{2}{e}$ are also eigenvectors of the metric tensor. Being concomitants of the metric, they also commute with the Killing vectors.

Let us check this for the Kerr space–time. Substituting explicit forms for the spin coefficients into (32) we get $\underset{1}{e}^i = (0, 2\Delta/\Sigma, 0, 0)$ and $\underset{2}{e}^i = (0, 0, (2ia \sin\theta)/\Sigma, 0)$. So $\underset{1}{\tilde{e}}^i = \frac{1}{2}(\Sigma/\Delta)\underset{1}{e}^i = (0, 1, 0, 0)$ and $\underset{2}{\tilde{e}}^i = (\Sigma/2ia \sin\theta)\underset{2}{e}^i = (0, 0, 1, 0)$ are eigenvectors of the Killing tensor and they also commute. In the general case it is more difficult to find the appropriate scaling factors.

In summary, we have shown that whenever vacuum type-D space–times possess a Killing tensor it is possible to construct two commuting Killing vectors from the Killing tensor and the metric and to choose two common eigenvectors of the metric and the Killing tensor so that they commute and are orthogonal to Killing vectors. The eigenvectors also commute with the Killing vectors. Therefore all the assumptions of theorem 1 are satisfied and we have the following theorem.

Theorem 2. A vacuum type-D space–time without acceleration possesses a regular S_2 separability structure. (It is completely determined by the Killing tensor and the metric.)

5. Conclusions

We have shown that in all vacuum type-D space–times without acceleration all the information necessary to establish separability of the geodesic equation can be obtained from the Killing tensor and the metric. Our result can be extended to the electrovac case.

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References

Benenti S 1975–6 *Rend. Sem. Mat. Univ. Pol. Torino* **34** 431
—— 1977 *Rep. Math. Phys.* **12** 311
Benenti S and Francaviglia M 1979 *J. Gen. Rel. Grav.* **10** 79
—— 1980 *Einstein Centennial Volume* ed. A Held (New York: Plenum)
Carter B 1969 *Commun. Math. Phys.* **10** 280
Havas P 1975 *J. Math. Phys.* **16** 1461
Hughston L P and Sommers P 1973a *Commun. Math. Phys.* **32** 147
—— 1973b *Commun. Math. Phys.* **33** 129
Hughston L P, Sommers P, Penrose R and Walker M 1972 *Commun. Math. Phys.* **27** 303
Kinnersley W 1969 *J. Math. Phys.* **10** 1195
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566
Penrose R 1968 *Battelle Rencontres, 1967* ed. C DeWitt and B DeWitt (New York: Benjamin)
Sommers P 1973 *PhD Thesis* University of Texas at Austin
Walker M and Penrose R 1970 *Commun. Math. Phys.* **18** 265
Woodhouse N M J 1975 *Commun. Math. Phys.* **44** 9